Thresholds for Quantum Computation

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Threshold

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Sounds like a number. Why are there so many?

Error model properties

Adversarial	The worst thing that can happen does happen.
Random	The error resulting from a fault is chosen randomly from the allowed set.
Local	Faults are associated with individual qubits or gates.
Coherent	Error amplitudes add.
Stochastic	Incoherent. Error probabilities add.
Heralded	Faults are announced.

...and many more, including Markovian, non-local, and systematic.

Engineering constraints

Available physical operations, e.g., linear optics

Measurement speeds, e.g., solid state

Which qubits can interact, e.g., atom lattices

Leakage of qubits, e.g., ion traps

Most frequently we assume none of these are problematic.

Of complexity classes and men

What we do not have

A general theory that tells us what component accuracy is necessary and sufficient to efficiently implement a quantum Turing machine.

What we do have

Explicit constructions for reducing the effective error to an arbitrarily small size

Quantum error correction

Quantum data cannot be directly inspected for error.

$$\alpha\,|001\rangle + \beta\,|110\rangle \xrightarrow{Z_1,Z_2,Z_3}{}_{\text{Measurement}}\ |001\rangle \ \text{or}\ |110\rangle$$

Errors are continuous.

$$(\sqrt{1-\delta^2}I + i\delta X_1)\,|000\rangle = \sqrt{1-\delta^2}\,|000\rangle + i\delta\,|100\rangle$$

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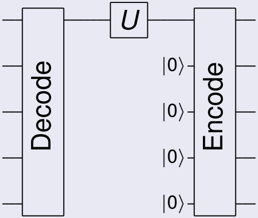
$$(\sqrt{1-\delta^2} I + i\delta X_1)|000\rangle = \sqrt{1-\delta^2}|000\rangle + i\delta|100\rangle$$

Use linearity of quantum mechanics, correct a basis.
Qubit basis: X, Y, and Z.

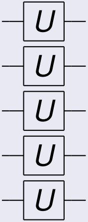
Encoded operations

Applying encoded U

Bad method



Good method

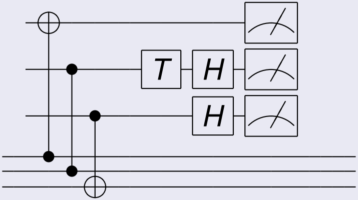


More generally, qubits in an encoded block should not interact.

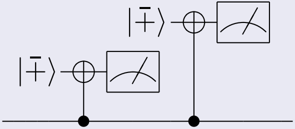
Fault tolerance A design strategy that seeks to minimize the spread of errors.

Fault-tolerance

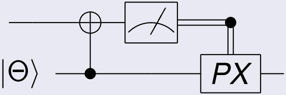
Transversal Gates



Repetition

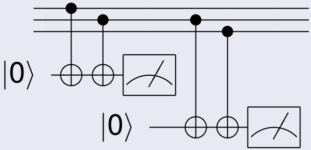


Teleportation



$$|\Theta\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \exp^{i\pi/4}|1\rangle)$$

Expenditure of Qubits



Discard on 1

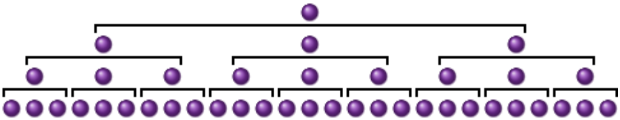
Concatenation

A fault-tolerant procedure maps unencoded gates and states to encoded ones with a new effective error rate.

The encoded error rate cannot be made arbitrarily small with a finite code.

Concatenated coding simply iterates the encoding map.

Each level of encoding decreases the effective error rate.



Estimating the threshold

Encoding does not always help. Error correction with unreliable components can make things worse.

Roughly, a fault-tolerant procedure is below threshold when

$$\left\{\begin{matrix}\text{Encoded}\\ \text{failure rate}\end{matrix}\right\} < \left\{\begin{matrix}\text{Unencoded}\\ \text{failure rate}\end{matrix}\right\},$$

The encoded failure rate can be estimated either analytically or by simulation.

Simulation and threshold estimation

Monte-Carlo simulation is simplified by two facts

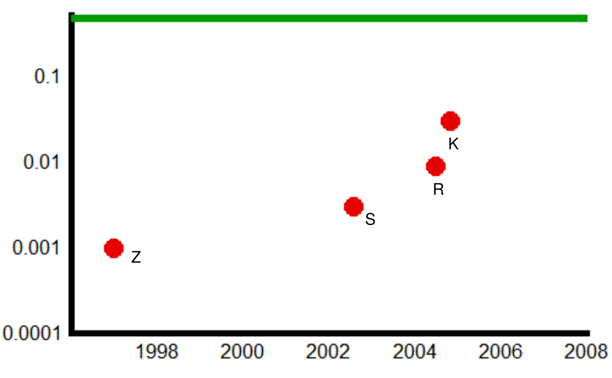
1. FT Procedures almost exclusively employ gates that preserve the Pauli group under conjugation.
2. Error checks project into the Pauli basis.

The first fact ensures that the impact of a Pauli error is easy to determine.

Together they imply that stochastic Pauli errors are an acceptable substitute for other stochastic channels.

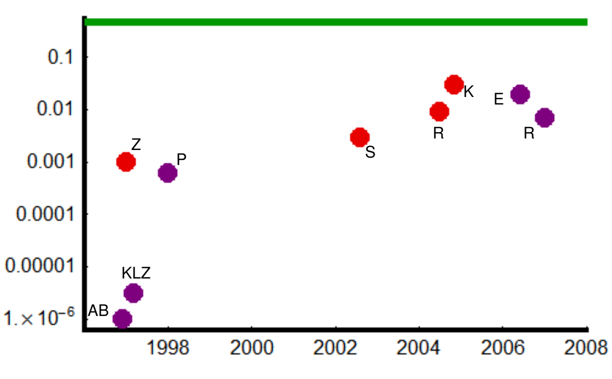
Thresholds through the ages

Numerical threshold estimates vs. date



Thresholds through the ages

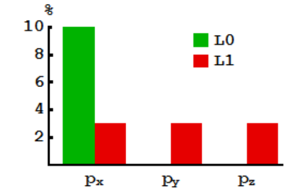
Numerical and analytical threshold estimates vs. date



Approximations

Two weaknesses exist in threshold arguments up till now.

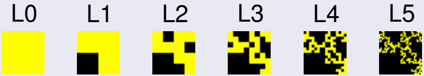
Example error model evolution



There are many kinds of errors on many kinds of gates. This complicates quantifying improvement.

The self-similarity of encoded versus unencoded qubits is not exact.

Worst-case good qubits



Rigorous threshold bounds

To obtain rigorous bounds, the mapping between encoded levels must be made precise.

Program following Aliferis et. q-ph/0504218

1. Specialize to codes that correct 1 error
2. Simplify the error model
3. Define "good" in a way that is recursive
4. Prove that gadgets with at most 1 fault cause no logical error
5. Count the number of ways that a procedure can fail

Adversarial local error model

Failures are assumed to strike components randomly.

The errors induced are assumed to be local to the faulty component, but otherwise adversarial.

Compared to the depolarizing channel

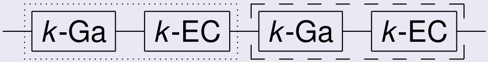
- Lowers threshold somewhat
  - Difficult to correct
  - Error is unphysical
- Eliminates the need to identify the encoded failure that results from an excess of component failures.

Good, bad, and rectangular

Rectangle definitions

k-Rec An encoded gate and the subsequent error corrections in a *k*th level circuit.

k-Recs



k-exRec An encoded gate and the preceding and subsequent error corrections in a *k*th level circuit.

k-exRecs

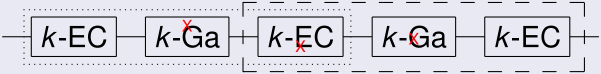
Good, bad, and rectangular

Bad A k-exRec is bad if it contains two independently bad $(k-1)$-exRecs.

Dependent A pair of bad k-exRecs are dependent if they overlap and the first k-exRec is not bad when the overlapping k-EC is ignored.

Good A k-exRec is good if it is not bad.

Dependent pair of bad k-exRecs

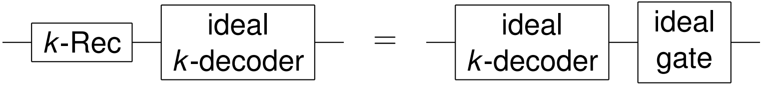


From these definitions it can be shown that a good k-exRec takes valid input blocks to valid output blocks.

Correct, as opposed to good

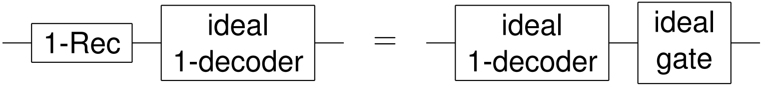
Ultimately, we care whether an encoded circuit gives the correct answer.

A correct k-Rec satisfies



Need to prove that good implies correct.

What we can prove is that, for a good 1-Rec



The threshold dance

— $(k+1)$-EC — $(k+1)$-Rec — $(k+1)$-decoder —

$=$ — $(k+1)$-EC — $(k+1)$-Rec — k-decoders — 1-decoder —

$=$ — k-decoders — 1-EC — 1-Rec — 1-decoder —

$=$ — $(k+1)$-EC — k-decoders — 1-decoder — 0-Ga —

$=$ — $(k+1)$-EC — $(k+1)$-decoder — 0-Ga —

The dance is performed with good k-Recs, good k-ECs, and ideal decoders. The resulting 0-Gates are perfect.

Bounding the threshold

Encoding is useful if a k-exRec is less likely to fail than a $(k-1)$-exRec.

To get a number for the threshold bound, set all failure probabilities equal.

For a procedure with at most g $(k-1)$-exRecs in a k-exRec the kth level failure rate satisfies

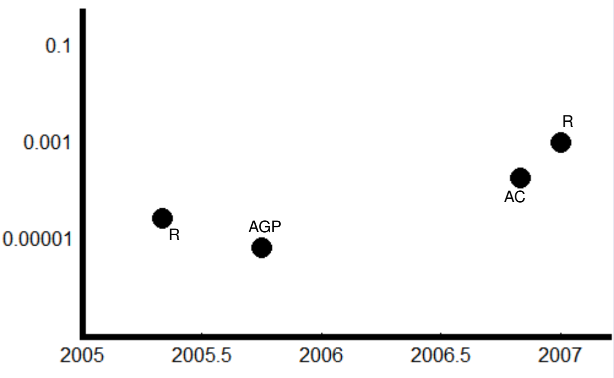
$$p^k < \binom{g}{2}(p^{(k-1)})^2.$$

A loose bound on the threshold is thus $\binom{g}{2}^{-1}$.

Better bounds can be obtained by adjusting for ancillae verification and counting only malignant pairs.

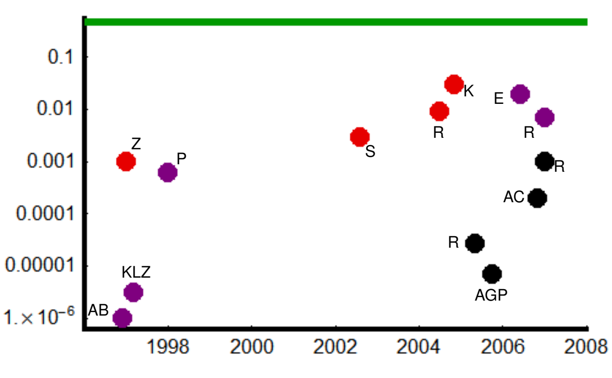
Thresholds through the ages

Threshold bounds vs. date



Thresholds through the ages

All threshold results vs. date



Leakage

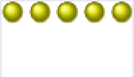
Heralded qubit loss represents no problem so long as new qubits can be inserted.

In other cases Knill's telecorrection method is useful. Knill q-ph/0410199

Standard error correction couples the data to an ancilla and the measures the ancilla.

Telecorrection teleports the data to an ancilla and measures the old data qubits.

Plugging leaks



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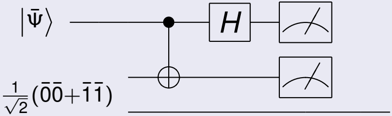

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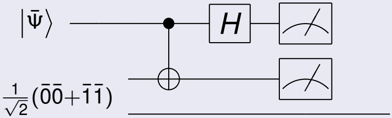

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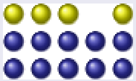

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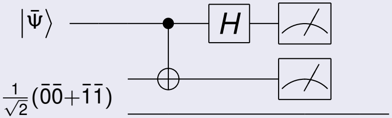

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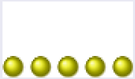

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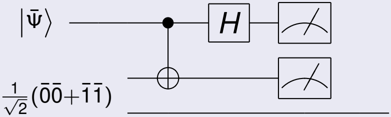


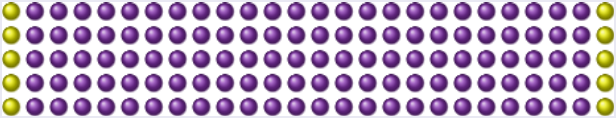
Local operations

Entanglement purification and teleportation can be used to create effectively non-local gates. Brennen et. q-ph/0301012

Svore et. have shown, however, that this is not necessary. Svore et. q-ph/0410199

They swap entire error correcting blocks to their destination, stopping along the way to error correct within the block.

Non-local gates for a local architecture



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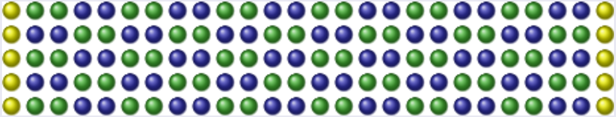

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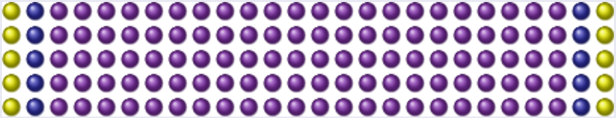

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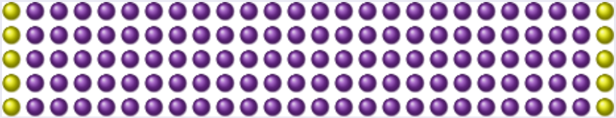

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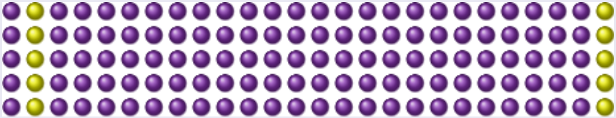

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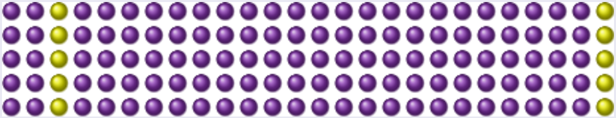

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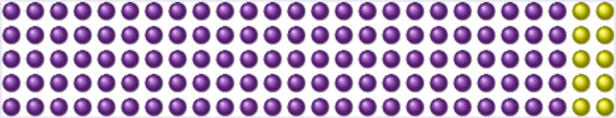


Slow measurements

Measurements are frequently used in threshold calculations.

Aharonov et.'s rigorous threshold bound used no measurements.

Instead, they applied error correction coherently.

More recently, DiVincenzo et. have shown that slow measurements have little effect on the threshold.

DiVincenzo et. q-ph/0607047

Divincenzo's solution to slow measurements

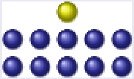
1. Error propagation - Error propagation allows correction to be postponed indefinitely.
2. Ancilla decoding - Ancilla verification is avoided using a method of ancilla measurement that exposes their communicable flaws.
3. Teleportation with state injection - State injection teleports an unencoded state to an encoded one. A gate is then applied to the data by teleporting it with the resulting state.

A word about state injection

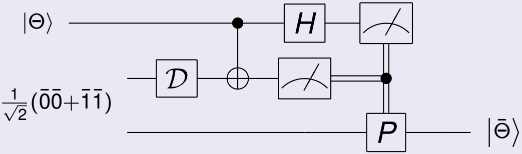
State injection

1. Prepare the encoded Bell state
$$\frac{1}{\sqrt{2}}(|\bar{0}\bar{0}\rangle + |\bar{1}\bar{1}\rangle).$$
2. Decode one of the pair to yield
$$\frac{1}{\sqrt{2}}(|0\bar{0}\rangle + |1\bar{1}\rangle).$$
3. Perform teleportation between an unencoded qubit and the half-encoded Bell pair.

State injection



Injection of $|\Theta\rangle$



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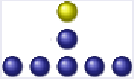

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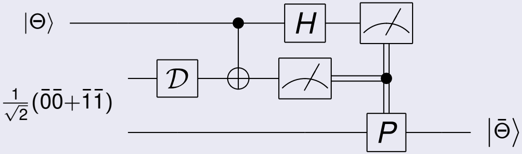

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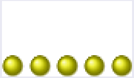

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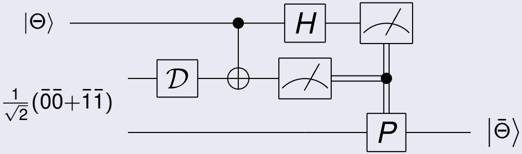

Linear-optics quantum computing

Neither ${}^{C}Z$ nor ${}^{C}X$ can be constructed using linear optics.

Measurement generates a kind of non-deterministic non-linearity.

Through state preparation and (also non-deterministic) teleportation it is possible to quantum compute.

Knill et. estimate a threshold of up to 50%!

But only for those errors.

Knill et. q-ph/0006120

Summary: Thresholds for all

Largest threshold estimate:	.03	Knill q-ph/0410199
Largest known threshold bound:	2×10^{-4}	Aliferis et. q-ph/0610063

Assumption dependence of the threshold

Change	Effect†	Source
Local gates	$\div 2$	Svore et. q-ph/0410199
Slow measurements	$\times 1$	DiVincenzo et. q-ph/0607047
Biased coherent errors	square	Preskill q-ph/9712048
Leakage	???	
LOQC	???	
Depolarizing errors	$\times 2$	Aliferis et. q-ph/0504218

† These effects are inferred, not proven.

A word of warning

The methods that produce the highest thresholds and those that deal with the most troublesome assumptions tend to require MANY more qubits.

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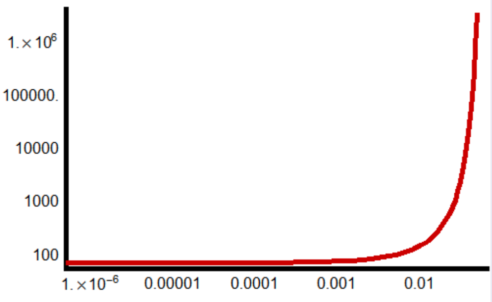
Guess-A-Plot: Scale-up vs. error probability (1 billion operations)



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P. Aliferis, D. Gottesman, and J. Preskill, *Quantum accuracy threshold for concatenated distance-3 codes*, quant-ph/0504218.

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Key to *Thresholds through the ages* graphs

- **AB** Aharonov and Ben-Or, {10^{-6},q-ph/9611025}
- **AC** Aliferis and Cross, {1.94×10^{-4},q-ph/0610063}
- **AGP** Aliferis, Gottesman, and Preskill, {2.73×10^{-5},q-ph/0504218}
- **E** Eastin, {.019,q-ph/0605192}
- **K** Knill, {.03,q-ph/0410199}
- **KLZ** Knill, Laflamme, and Zurek, {3×10^{-6},q-ph/9702058}
- **P** Preskill, {6×10^{-4},q-ph/9712048}
- **R** Reichardt, {.009,q-ph/0406025}, {.007,q-ph/0612004}, {6.75×10^{-6},q-ph/0509203}, {.001,q-ph/0612004}
- **S** Steane, {.003,q-ph/0207119}
- **Z** Zalka, {.001,q-ph/9612028}