Restrictions on Transversal Encoded Quantum Gate Sets

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Motivation

Transversal encoded gates are inherently fault tolerant.

Desired

A quantum code with a universal, transversal encoded gate set

Universal, transversal encoded gate sets are hard to find.

Alternate desire

A proof that such gate sets don't exist

Previous work

- Qubit stabilizer codes:
  Zeng, Cross, and Chuang, arXiv:0706.1382
- Qudit stabilizer codes:
  Chen, Chung, Cross, Zeng, and Chuang, arXiv:0801.2360

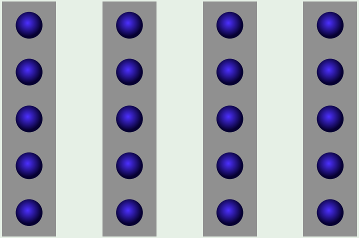
Definitions

Quantum code Subspace of Hilbert space, defined by projector, P

Detectable error Error E satisfying $PEP \propto P$

Code block Unit of independent error detection

Code with four code blocks



Transversal partition Partition such that each part contains one subsystem from each code block

Transversal operator Operator which only couples subsystems within a transversal partition

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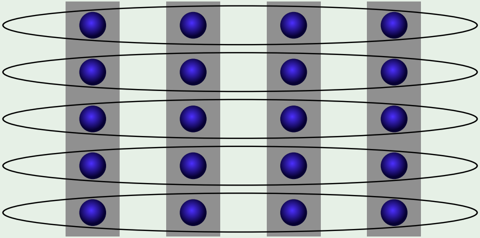

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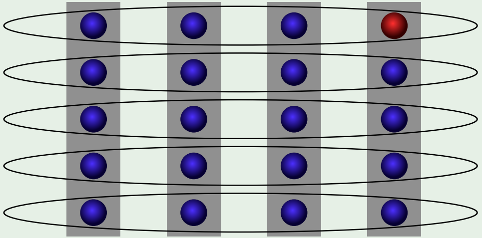

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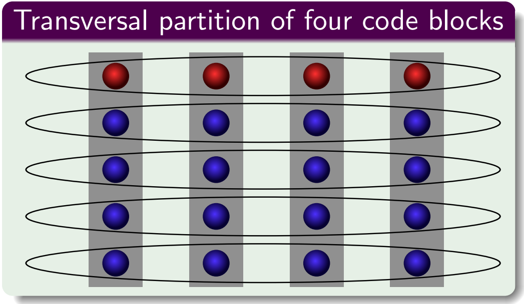

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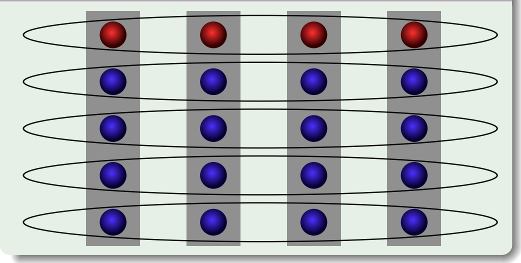

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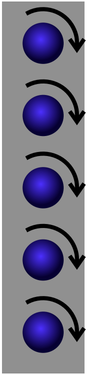
Transversal operator Operator which only couples subsystems within a transversal partition

Product operator Operator which does not couple subsystems

Inspiration

Consider a code with logical gates of the form

$$\overline{R(Z,\theta)} = \mathrm{e}^{\mathrm{i}\theta Z} \otimes \mathrm{e}^{\mathrm{i}\theta Z} \otimes \cdots \otimes \mathrm{e}^{\mathrm{i}\theta Z} = \prod_j \mathrm{e}^{\mathrm{i}\theta Z_j} .$$



For small θ,

$$\overline{R(Z,\theta)} \approx \prod_j (I + \mathrm{i}\theta Z_j) \approx I + \mathrm{i}\theta \sum_j Z_j .$$

Z_j must be a logical gate. Else, $\overline{R(Z,\theta)}$ would not be one.

Conclusion: Such codes cannot be local-error-detecting.

Inspiration Issues

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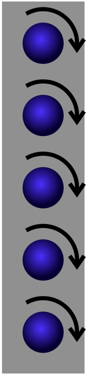
The transversal logical gates need not have this form.



For small θ,

$$\overline{R(Z,\theta)} \approx \prod_j (I + \mathrm{i}\theta Z_j) \approx I + \mathrm{i}\theta \sum_j Z_j \ .$$

Requires continuity.

Z_j must be a logical gate. Else, $\overline{R(Z,\theta)}$ would not be one.
True if the code is additive.

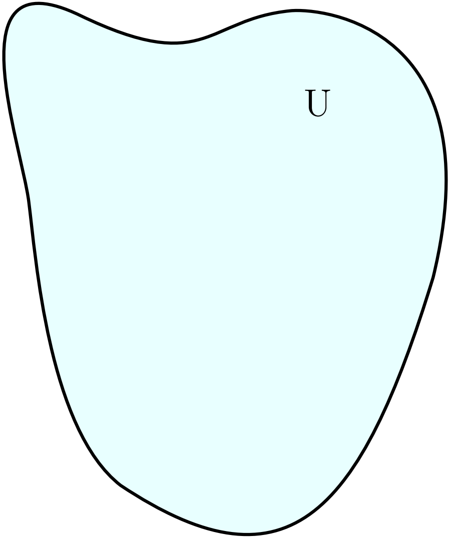
Conclusion: Such codes cannot be local-error-detecting.

Outline of the Proof

Steps

1. The logical product operators form a Lie subgroup G.
2. G partitions into a finite number of cosets.
3. Each coset of G yields one logically distinct operator.
4. A finite set of operators is not universal. Product operators are not universal.
5. The transversal logical operators are not universal.

The logical product operators are a Lie subgroup



U - Unitary operators

T - Product operators, T such that

$$T = \bigotimes_j U_j$$

L - Logical operators, L such that

$$(I - P)LP = 0$$

$\mathrm{G} = \mathrm{L} \cap \mathrm{T}$ - Logical product operators

G is a Lie subgroup.

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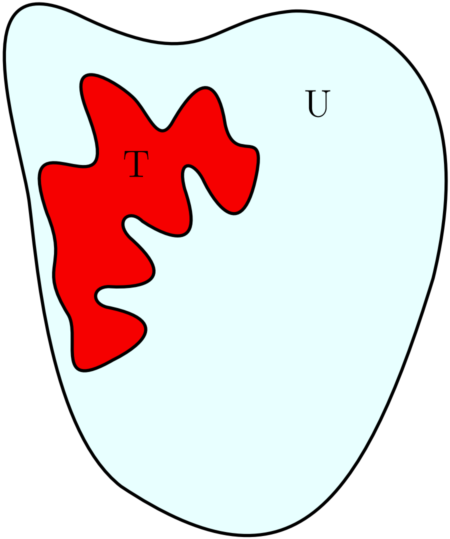

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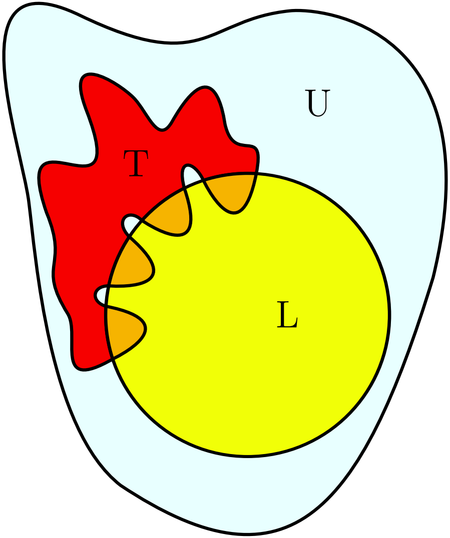

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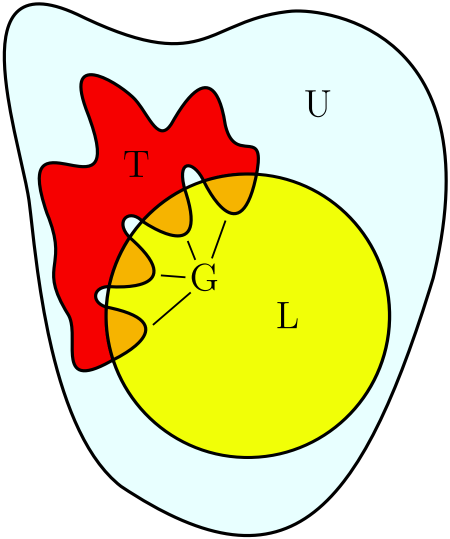

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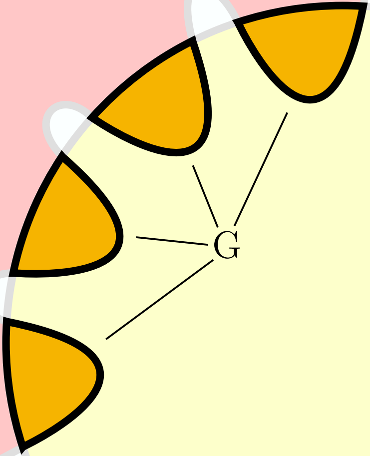
Partitioning the logical product operators



Because G is a Lie group,

- G can be partitioned into connected components
- one is the connected component of the identity, C, and a Lie group
- all other components are cosets of C

F - Set of representatives of the cosets

The cosets are

- discrete
- finite in number

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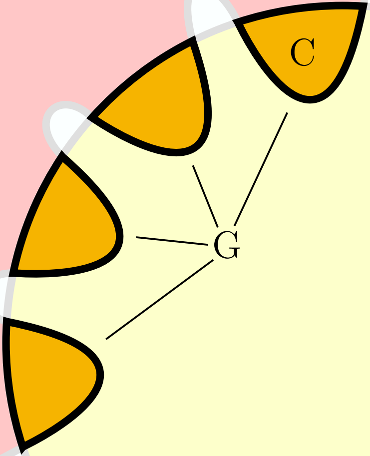

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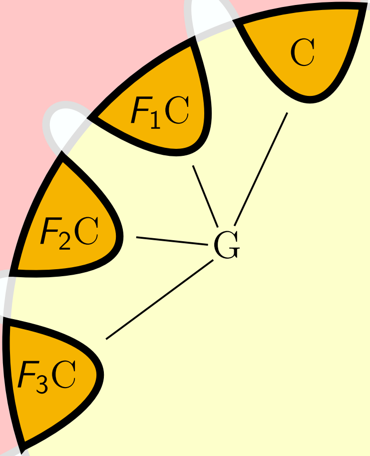

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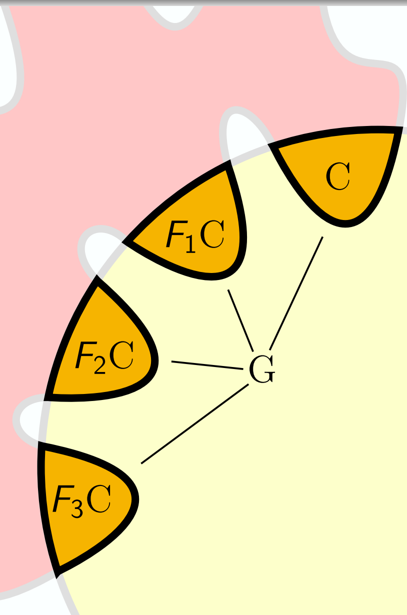


The component of the identity, C,

- is a Lie subgroup of T
- has a Lie algebra of sums of local, Hermitian operators

A local, Hermitian operator H satisfies the local-error-detection condition:

$$PHP \propto P.$$

Elements of the Lie algebra of C

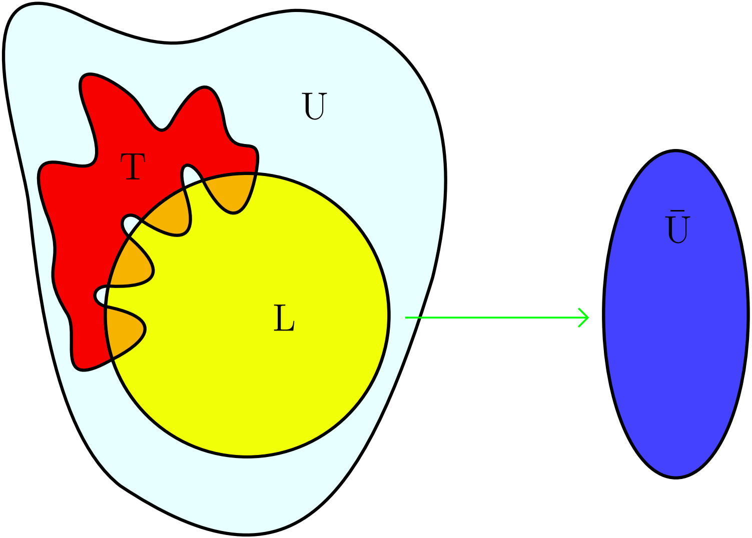
- act trivially on the code space
- are logical operators

Elements in C implement logical identity.

F indexes the logically distinct gates in G.

A finite number of operators is not universal

Consider the space of operators on the encoded system.



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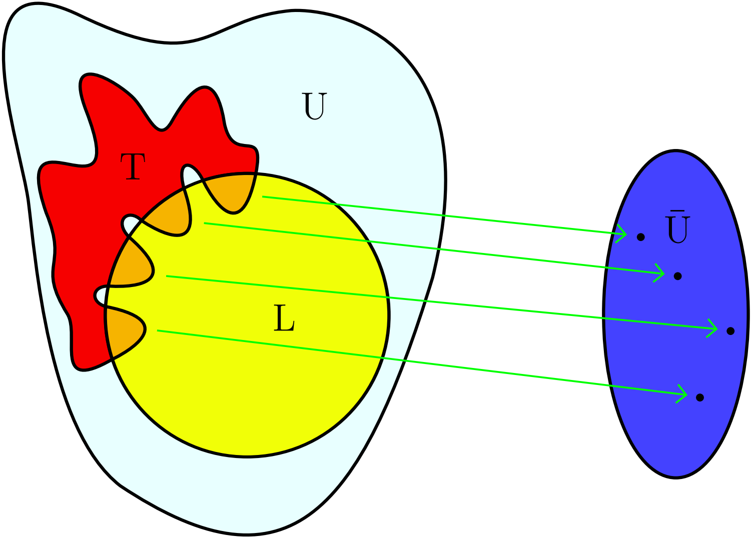


A finite number of operators is not universal

Consider the space of operators on the encoded system.

The logically distinct operators in G are

- represented by F
- finite in number

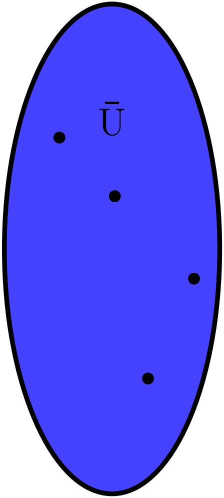
Desired set of logically distinct operators is infinite.

Arbitrary approximation is impossible.

Theorem 1

A local-error-detecting code cannot have a universal set of product operators.

Contrasts with a finite **basis** of gates



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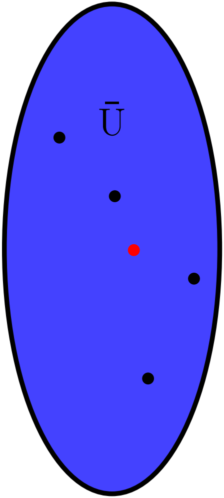

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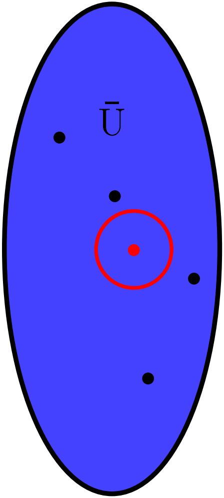

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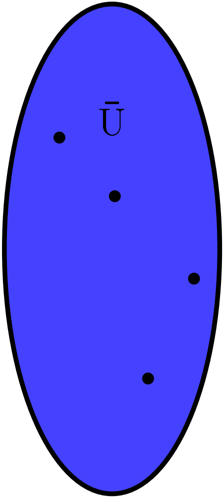

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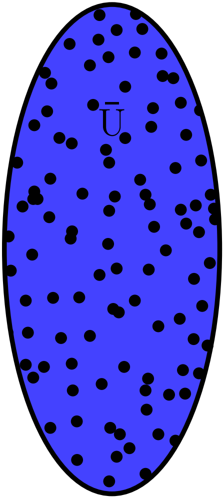

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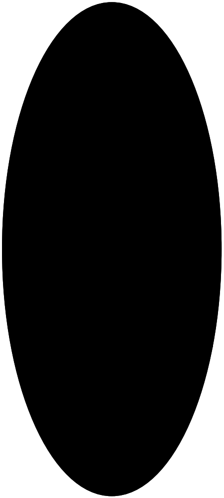
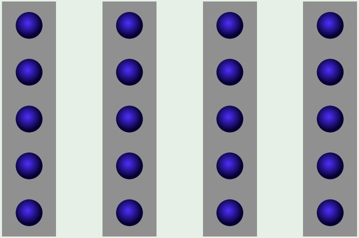


The transversal logical operators are not universal

Each transversal part satisfies the detection property.

Subsystems of a code



Apply Theorem 1 to the transversal parts.

Corollary 1

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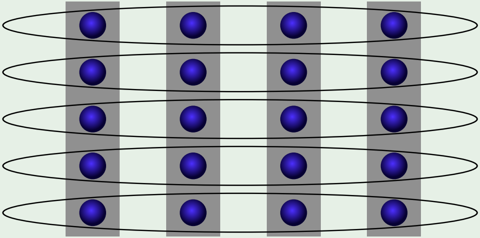
Transversal parts of a code



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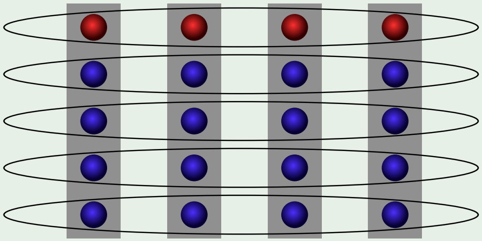
Error on a transversal part



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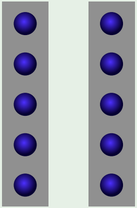
Circumventions

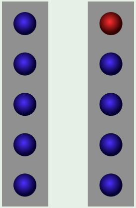
Non-unitary operators



E.g., measurement and classical feed forward

Requires ancillae

Stepwise-transversal operators

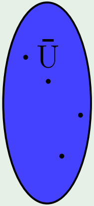


Switch partitions

Error correct in between

Universality?

Approximate universality

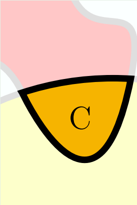


Finite set of logically distinct gates

Non-universal

Accuracy scaling?

Probabilistic error detection



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Transversality & universality?

Failure rate scaling?

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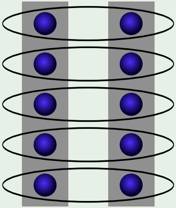

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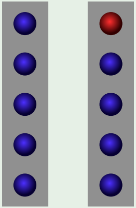

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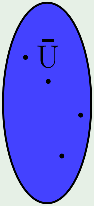

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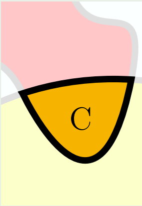

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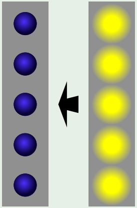

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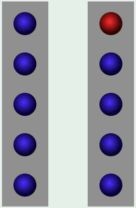

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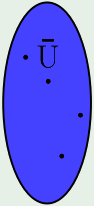

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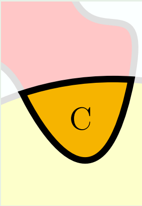

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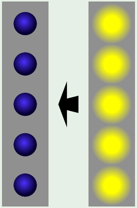

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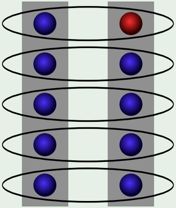

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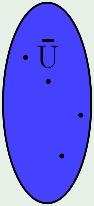

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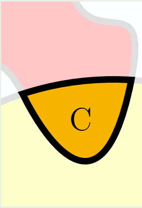

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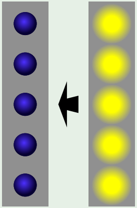

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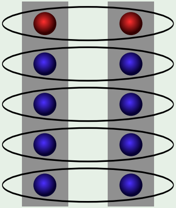

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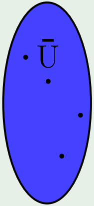

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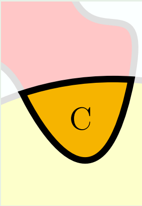

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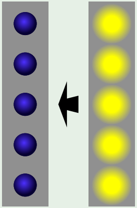

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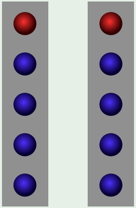
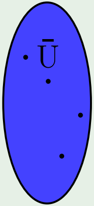


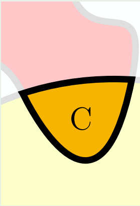
- Switch partitions
- Error correct in between
- Universality?

Approximate universality



- Finite set of logically distinct gates
- Non-universal
- Accuracy scaling?

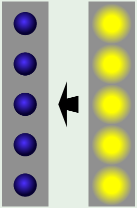
Probabilistic error detection



- $PEP \propto P$
- Transversality & universality?
- Failure rate scaling?

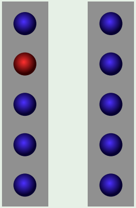
Circumventions

Non-unitary operators



- E.g., measurement and classical feed forward
- Requires ancillae

Stepwise-transversal operators



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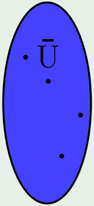

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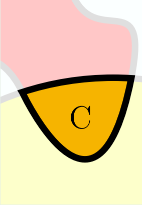

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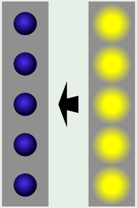

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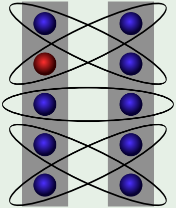

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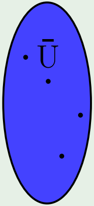

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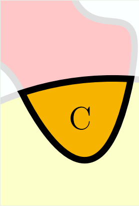


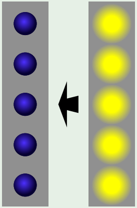
PEP $\propto$ *P*

Transversality & universality?

Failure rate scaling?

Circumventions

Non-unitary operators



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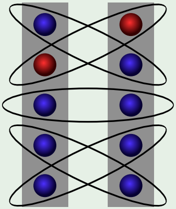

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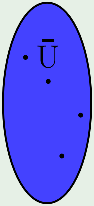

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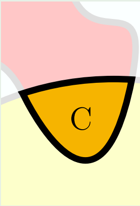

$PEP \propto P$

Transversality & universality?

Failure rate scaling?

Conclusion

Result

A local-error-detecting code cannot have a universal set of transversal operators.

Circumventions

- Non-unitary operators
- Stepwise transversal operators
- Approximate universality
- Probabilistic error detection

References

- Eastin and Knill, arXiv:0811.4262
- Zeng, Cross, and Chuang, arXiv:0706.1382
- Chen, Chung, Cross, Zeng, and Chuang, arXiv:0801.2360